"SEQUENCE"

① Def^n: Sequence:

A f^n $S : \mathbb{N} \to \mathbb{R}$ is called seq^n or real seq.

denoted by $\{S_n\}, n \in \mathbb{N}$

$\{S_n\} = \{S_1, S_2, S_3, \ldots, S_n, \ldots\}$

e.g. 1. $\{S_n\} = \{(-1)^n\}$

$= \{-1, 1, -1, 1, -1, 1 \ldots\}$

2. $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$

② Range of seq^n: $\boxed{\text{Range set}}$

The set of all distinct element of seq. is called range of seq.

e.g. ① Range = $\{-1, 1\}$

② Range = $\{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$

③ Bounded above seq:

$\{S_n\}$ is said to bdd above if $\exists\, k_1 \in \mathbb{R} \ni S_n \leq k_1, \forall n$

upper bound

③ $\{S_n\}$ is bdd below if
$\exists\ k_2 \in R \ni S_n \geq k_2, \forall n$
lower bound

④ Bounded seqn.

$\{S_n\}$ is said to bounded iff it is bdd above as well as bdd. below.

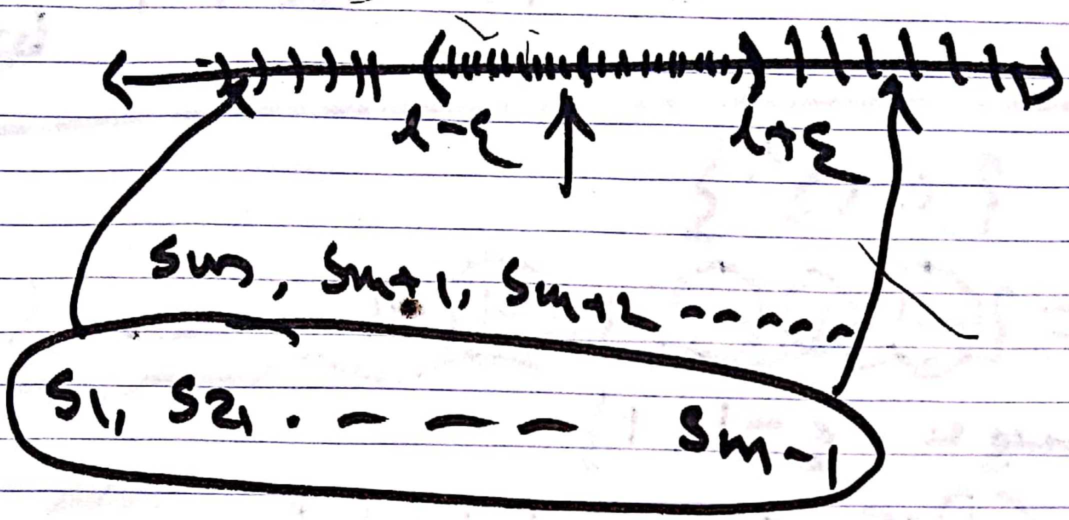
Remark. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$
$-1 \leq S_n \leq 1$
Range set $= \{-1, 1\}$

① seq is bdd iff its range is bdd.

⑤ Convergence of seqn:

$\{S_n\}$ converge to real no l, if for each $\varepsilon > 0\ \exists\ m \in N \ni$
$\Rightarrow |S_n - l| < \varepsilon, \forall n \geq m$
$\Rightarrow l - \varepsilon < S_n < l + \varepsilon, \forall n \geq m$
$\Rightarrow S_n \in (l - \varepsilon, l + \varepsilon), \forall n \geq m$

$s_n \in (l-\varepsilon,\ l+\varepsilon) \quad \forall\ \boxed{n \geq m}$



s_n converges to l is denoted by $\boxed{\lim_{n \to \infty} s_n = l}$

⑥ **limit of seqn:**

A real no. ξ is said to be limit pt. of $\{s_n\}$, if every nbhd of ξ contains infinitely many no. of members of seq.

$n \to \infty\ s_n = \xi$

for $\varepsilon > 0$ $\exists$

$|s_n - \xi| < \varepsilon$ for infinitely many value of n

Ex. / e.g.

(1) Is seq. bdd
→ Is it conv.
→ Find limit
→ Find upper bdd & lower bdd

→ Find Range seqn

(1) $\{(-1)^n\}$

Soln

$= \{-1, 1, -1, 1, -1, 1, \ldots\}$

Range $= \{-1, 1\}$

$-1 \leq S_n \leq 1$ bdd ✓

l.b u.b

seq. is not cgt.

$S_1 = S_3 = S_5 = S_7 = \ldots -1$

$S_1, S_3, S_5, \ldots$

$-1-\varepsilon$

$\boxed{-1 \text{ is limit pt. of } S_n}$

$S_2 = S_4 = S_6 = \ldots = 1$

$\boxed{1 \text{ is limit pt of seq.}}$

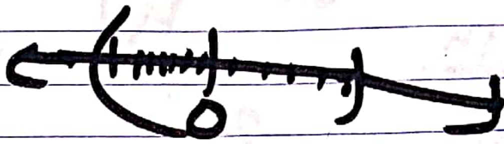
2.

(2) $\{1, 2, 3 \ldots n, 1, 2, 3 \ldots n, 1, 2, 3 \ldots n \ldots\ldots\}$

soln. $1 \le S_n \le n$

(3) $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \frac{1}{n}, \ldots\}$

$S_n \longrightarrow 0$



limit of $S_n = 0$

$\boxed{0 < S_n \le 1}$

(4) $\{1 + (-1)^n\} = \{0, 2, 0, 2, \ldots\}$

Range set $= \{0, 2\}$

$0 \le S_n \le 2$

$S_n \longrightarrow$ is not cgt.

$\boxed{n \to \infty \; S_n = 0}$ $\boxed{n \to \infty \; S_n = 2}$

limit pt. of $S_n = \{0, 2\}$

(5) $\{(-1)^n (1 + \frac{1}{n})\} = \{-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4},$

$-2, -\frac{4}{3}, -\frac{6}{5}, -\frac{8}{7}, \ldots\ldots \to -1$ $\quad -\frac{6}{5}, \frac{7}{6}, \ldots\ldots\}$

$\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10} \ldots\ldots \to 1$

$\boxed{-2 \le S_n \le} \; \frac{3}{2} < 2 < 1$

limit pt. of set $\{-1, 1\}$

Find limit pt. of range set

$R = \{$

Ex. $\left\{ \frac{(-1)^{n-1}}{n!} \right\}$

soln. $= \left\{ 1, \frac{-1}{2!}, \frac{1}{3!}, \frac{-1}{4!}, \frac{1}{5!}, \frac{-1}{6!}, \ldots \right\}$

soln.

→ Range $= \{$

→ bound $= -\frac{1}{2} \leq S_n \leq 1, \forall n$

→ ~~cgt.~~ limit pt. $\boxed{0}$

→ cgt : $\boxed{0}$ $S_n \longrightarrow 0$

→ limit pt. of range set $\boxed{0}$

Hw $\{n^2\} = \{1, 4, 9, 16 \ldots\}$

$1 \leq S_n < \infty$

Divergence seq:

A seq. which is not cvt. is called <u>divergence seq</u>

(i) <u>Oscillate finitely</u>:-

A seq. is said to be finitely Oscillate if seq. is bounded & it has atleast 2 limit pts.

(ii) <u>Diverges to $-\infty$</u>:-

A seq. is said to be diverges to $-\infty$, if seq. is bdd above but not bdd below and seq. has no other limit pt besides $-\infty$.

(iii) <u>Diverges to $+\infty$</u>!

A seq is said to be diverges to $+\infty$, if seq. is bdd below but not bdd above and seq. has no other limit pt <u>besides $+\infty$</u>

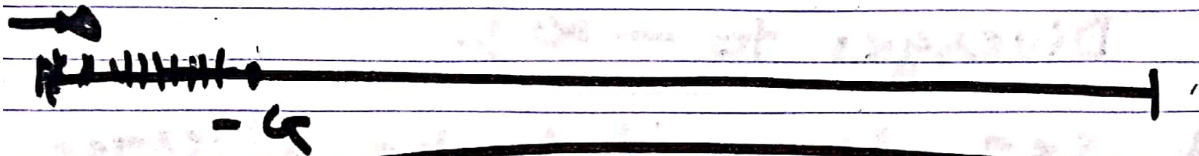
(iv) <u>Oscillate infinitely</u>:

A ~~seq~~. unbounded seq. is said to be oscillate infinitely if it diverges neither to $+\infty$ nor to $-\infty$.

Def^n:

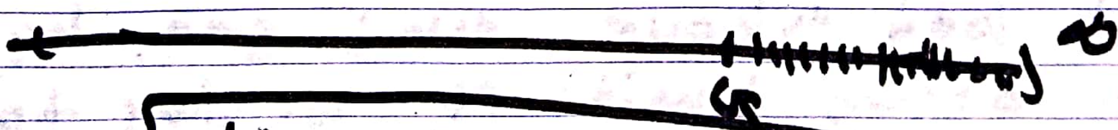
① If $\{S_n\}$ is not bdd below seq. we say that $S_n \to -\infty$ (i.e. $-\infty$ is limit pt. of S_n) if for any $G > 0$, however large $\exists$ +ve integer $m \ni$

$$S_n < -G \quad \forall\, n \geq m$$



i.e. $\boxed{\liminf_{n \to \infty} S_n = -\infty}$

② If $\{S_n\}$ is not bdd above we say that $S_n \to +\infty$, if for any $G > 0$, however large, $\exists\, m \in \mathbb{N} \ni S_n > G,\ \forall\, n \geq m$



i.e. $\boxed{\limsup_{n \to \infty} S_n = +\infty}$

EX: Examine following seq. for Convergent or Divergent.

① $\{1+(-1)^n\}$ $\overset{Sol^n}{=} \{0, 2, 0, 2, \ldots\}$

Here $0 \le S_n \le 2$ bdd seq.
limit pt. are 0 & 2
$\therefore$ seq is oscillate finitely

② $\{(-1)^n(1+\frac{1}{n})\} = \{-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, \ldots\}$

Here $-2 \le S_n \le \frac{3}{2}$, bdd seq.
Also limit pts are -1 & 1
$\therefore$ Oscillate finitely

③ $\{n^2\}$ sol^n $= \{1, 4, 9, 16, 25, \ldots\}$

Here $1 \le S_n < \infty$ not bdd above & it is bdd below
Also $\lim_{n \to \infty} S_n = +\infty$
$\therefore$ seq. diverges to $+\infty$

④ $\{-2^n\} = \{-2, -4, -8, -16, \ldots\}$

Here $-\infty < S_n \le -2$ bdd above but not bdd below
Also $n \to \infty$ $S_n = -\infty$ $\therefore$ $\boxed{\text{div. to } -\infty}$

⑤ $\{n(-1)^n\} = \{-1, 2, -3, 4, -5, 6, \ldots\}$

Here $-\infty < S_n < \infty$
it is not bdd below & above
$\therefore$ Oscillate infinitely

6) $\left\{\frac{(-1)^{n-1}}{n!}\right\} = \left\{1, \frac{-1}{2!}, \frac{1}{3!}, \frac{-1}{4!}, \frac{1}{5!}, \cdots\right\}$

Here $-\frac{1}{2} \leq S_n \leq 1$

Also $\lim_{n \to \infty} S_n = 0$

$\therefore$ 0 is only limit pt.

$\therefore$ Seq. is convergent to 0

7) $\left\{1+\frac{1}{n}\right\} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots\cdots\right\}$

Here $1 < S_n \leq 2$, Also $\boxed{\lim_{n \to \infty} S_n = 1}$

$\therefore$ 1 is limit pt. convergent

$\therefore$ $\boxed{\text{conv to } 1}$

8) $\left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, \cdots\cdots\right\}$

$0 < S_n < \infty$

it is bdd below but not bdd above

Also $\lim_{n \to \infty} S_n = 0$

& $\lim_{n \to \infty} S_n = +\infty$

$\therefore$ seq. is oscillate infinitely

9) $\{S_n\}$, where $S_n = \begin{cases} 2, & \text{when } n \text{ is even} \\ \text{lowest prime factor } (\neq 1) \text{ of } n, & \text{when } n \text{ is odd} \end{cases}$

Sol^n. $= \left\{ \begin{array}{l} 2, 2, 2, \ldots 2 \ldots \\ 3, 5, 7, 3, 11, 13, \\ 3, 17, 19, 3, 23, 5, 3, 29, 31, 3 \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\}$

Here $2 \leq S_n < \infty$ Not bdd.

limit pt. are : $2, 3, 5, 7, 11, 13, \ldots\ldots$

$\therefore$ seq. is Oscillate infinitely

$$\left. \begin{array}{l} \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10} \ldots \\ \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots\ldots \end{array} \right\} \longrightarrow 1$$

10) $\left\{ 1 + \frac{(-1)^n}{n} \right\} = \left\{ 0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \ldots \ldots \right\}$

$0 \leq S_n \leq \frac{3}{2}$

$\boxed{n \to \infty \; S_n = 1}$ $\therefore$ 1 is only limit pt

$\therefore$ seq. converges to 1

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10) $\left\{ m+\frac{1}{n} \right\}$, $m, n \in \mathbb{N}$ | $m=1,$ $n=1,2,\ldots$

$= \{ 1+1=2, 1+\frac{1}{2}=\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

$2+1=3, \frac{5}{2}, \frac{7}{3}, \ldots$

$3+1=4, \frac{7}{2}, \frac{10}{3}, \ldots$

$- - - - - - - - \}$

limit pt. are $= 1, 2, 3, 4 \ldots$

Oscillate infinitely

11) $\{ n+(-1)^n n \}$

$= \{0, 4, 0, 8, 0, 12, 0, 16, 0, 20, \ldots\}$

Here $0 \leq S_n < \infty$ bdd below but not above

$\liminf_{n\to\infty} S_n = 0$ & $\lim_{n\to\infty} S_n = +\infty$

$\therefore$ seq. is Oscillate infinitely

12) $\{(-1)^n\} = \{-1, 1, -1, +1, \ldots\}$

$-1 \leq S_n \leq 1$ bdd & limits $= -1, 1$

Oscillate finitely

(5)

Ex: By using definition prove the following

~~(1) $n \to \infty \; \sqrt[n]{n} = 1$~~

(2) $\lim_{n \to \infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$

sol: we have to p.T

For any $\varepsilon > 0 \; \exists \; m \in N \ni$

$$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon \quad \forall \boxed{n \geq m}$$

W.K.T $\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right|$

$$= \left| \frac{3+\cancel{2\sqrt{n}}-\cancel{2\sqrt{n}}}{\sqrt{n}} \right| = \left| \frac{3}{\sqrt{n}} \right| < \varepsilon,$$

if $\frac{3}{\sqrt{n}} < \varepsilon$

if $\frac{3}{\varepsilon} < \sqrt{n}$

if $\frac{9}{\varepsilon^2} < n$ if $\boxed{n > \frac{9}{\varepsilon^2}}$

Let $m \in N$ greater than $\frac{9}{\varepsilon^2}$

i.e. $m > \frac{9}{\varepsilon^2}$.

$$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon \quad \forall n \geq m$$

2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Let $\sqrt[n]{n} = 1 + h_n$, where $h_n \geq 0$

then $n = [1+h_n]^n$

$\Rightarrow n = 1 + {}^nC_1 h_n + {}^nC_2 h_n^2 + \dots + h_n^n$

$\Rightarrow n = 1 + n h_n + \underbrace{\frac{n(n-1)}{2!} h_n^2}_{} + \dots + h_n^n$

$\Rightarrow \cancel{n} > \frac{\cancel{n}(n-1)}{2!} h_n^2 \quad \forall n$

$\Rightarrow h_n^2 < \frac{2}{n-1} \quad \forall n \geq 2$

$\Rightarrow |h_n| < \sqrt{\frac{2}{n-1}} \quad \forall n \geq 2$

$\Rightarrow |\sqrt[n]{n} - 1| < \sqrt{\frac{2}{n-1}} \quad \forall n \geq 2$

$\Rightarrow \qquad < \varepsilon$

Let $m \in N$ & $m > \frac{2}{\varepsilon^2} + 1$ Then $\lim_{n\to\infty} \sqrt[n]{n} = 1$	if $\sqrt{\frac{2}{n-1}} < \varepsilon$
	if $\frac{2}{n-1} < \varepsilon^2$
	if $\frac{2}{\varepsilon^2} + 1 < n$
	ie $n > \frac{2}{\varepsilon^2} + 1$

3) $\lim_{n \to \infty} \frac{2n-3}{n+1} = 2$

Sol^n. $\left| \frac{2n-3}{n+1} - 2 \right|$

$= \left| \frac{2n-3-2n-2}{n+1} \right|$

$= \left| \frac{-5}{n+1} \right| < \varepsilon$, if $\frac{5}{n+1} < \varepsilon$

if $\frac{5}{\varepsilon} < n+1$

if $\frac{5}{\varepsilon} - 1 < n$

if $\boxed{n > \frac{5}{\varepsilon} - 1}$

Let $m \in \mathbb{N} \Rightarrow m > \frac{5}{\varepsilon} - 1$

Thus $\left| \frac{2n-3}{n+1} - 2 \right| < \varepsilon$, $\forall\, n \geq m$

$\therefore \boxed{\lim_{n \to \infty} \frac{2n-3}{n+1} = 2}$

4) $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2} = \frac{1}{2}$

Solⁿ $\left| \frac{1+2+3+\cdots+n}{n^2} - \frac{1}{2} \right|$

$= \left| \frac{n(n+1)}{2n^2} - \frac{1}{2} \right|$

$= \left| \frac{n^2+n-n^2}{2n^2} \right|$

$= \left| \frac{1}{2n} \right| < \varepsilon$, if $\frac{1}{2\varepsilon} < n$

i.e if $n > \frac{1}{2\varepsilon}$

Let $m \in \mathbb{N} \ni m > \frac{1}{2\varepsilon}$

Then

$\left| \frac{1+2+\cdots n}{n^2} - \frac{1}{2} \right| < \varepsilon, \ \forall\, n \geq m$

$\therefore \lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \frac{1}{2}$

5) $\lim_{n \to \infty} \frac{1+3+5+\dots+(2n-1)}{n^2} = 1$ ⑥

Sol^n

$$\left| \frac{\sum (2n-1)}{n^2} - 1 \right|$$

$$= \left| \frac{2\sum n - \sum 1}{n^2} - 1 \right|$$

$$= \left| \frac{\cancel{2}\frac{n(n+1)}{\cancel{2}} - n}{n^2} - 1 \right|$$

$$= \left| \frac{n^2+n-n}{n^2} - 1 \right|$$

$$= |1-1|$$

$= \underline{0 < \varepsilon}$. Thus For any $m \in N$

~~Thus Let~~ ~~$m > \varepsilon$~~

we get

$$\left| \frac{\sum (2n-1)}{n^2} - 1 \right| < \varepsilon \quad \forall n \geq m$$

Ex 2 (continue) i.e $n > \frac{2}{\varepsilon^2} + 1$

Let $m \in N$ & $m > \frac{2}{\varepsilon^2} + 1$

Then

$$|\sqrt[n]{n} - 1| < \varepsilon \quad \forall n \geq m$$

$\Rightarrow \lim_{n \to \infty} \sqrt[n]{n} = 1$

Ex: Find $\lim_{n \to \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}}$

$$= \lim_{n \to \infty} \frac{\cancel{\sqrt{n}} \left[\frac{3}{\sqrt{n}} + 2 \right]}{\cancel{\sqrt{n}}}$$

$= 0 + 2$

$= 2$

$=$ RHS

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Theorem: 1 Prove that every cgt. sequence is bounded

Proof: Let $\{S_n\}$ be any seq. converges to l, then by definition we say that for any $\varepsilon > 0$ $\exists$ +ve integer $m \ni$

$$|S_n - l| < \varepsilon \quad \forall \; n \geq m$$

$$\Rightarrow \quad -\varepsilon < S_n - l < \varepsilon$$

$$\Rightarrow \quad l - \varepsilon < S_n < l + \varepsilon \quad \forall \; n \geq m \quad \text{--- (1)}$$

Let $g = \min\{l-\varepsilon, S_1, S_2, \ldots, S_{m-1}\}$

$G = \max\{l+\varepsilon, S_1, S_2, \ldots, S_{m-1}\}$

Then
$$\left.\begin{array}{l} g \leq S_i \quad \forall \; i = 1, 2, \ldots, m-1 \\ G \geq S_i \quad \forall \; i = 1, 2, \ldots m-1 \\ \text{Also } g \leq l - \varepsilon \;\&\; G \geq l + \varepsilon \end{array}\right\} \text{--- (2)}$$

Thus by (1) & (2), we say that

$$g \leq l - \varepsilon < S_n < l + \varepsilon \leq G \quad \forall n$$

ie $\quad g \leq S_n \leq G \quad \forall n$

Hence $\{S_n\}$ is bounded

Theorem: 2 P.T Every cgt. seq. has a unique limit

OR P.T. a seq. can not converge to more than one limit

Proof: suppose seq. $\{S_n\}$ converge to two distinct limit l & l'

$\therefore$ by def^n, we say that for any $\varepsilon > 0$ $\exists$ $m_1, m_2 \in N \ni$

$|S_n - l| < \varepsilon/2 \quad \forall\ n \geq m_1$

$|S_n - l'| < \varepsilon/2 \quad \forall\ n \geq m_2$

Let $m = \max\{m_1, m_2\}$ then

$m \geq m_1$ & $m \geq m_2$

Also $|l - l'| = |\underbrace{l - S_n} + \underbrace{S_n - l'}|$

$\leq |l - S_n| + |S_n - l'|$

$< \varepsilon/2 + \varepsilon/2, \quad \forall\ n \geq m$

$= \varepsilon, \quad \forall\ n \geq m$

$\Rightarrow l = l'$

Theorem. 3 Prove that every cgt. sequence is bounded and has a unique limit

Proof: Write proof of above theorem I & 2.

Theorem-4 State and Prove Bolzano-Weierstrass Theorem

Statement: P.T. Every bounded sequence has a limit point.

Proof:

Let $\{S_n\}$ be any bdd seq. and $S = \{S_n : n \in N\}$ be its range.

Since $\{S_n\}$ is bdd $\therefore S$ is bdd set.

Case-(i) If S is finite, then

$\exists$ atleast one $\xi \in S$ such that $S_n = \xi$ for an infinite no. of values of n.

$\Rightarrow S_n \in (\xi - \varepsilon, \xi + \varepsilon)$ for infinite no. of values of n. (for any $\varepsilon > 0$)

$\Rightarrow$ ξ is a limit pt. of $\{S_n\}$ ($\because S_n = \xi$)

Case (ii) If S is infinite, then S is infinite bounded set

$\therefore$ By Bolzano-Weierstrass thm for sets, we say that S has atleast one limit pt, say ξ

$\therefore$ every nbhd $(\xi - \varepsilon, \xi + \varepsilon)$ of ξ contains infinitly many member of S

$\therefore S_n \in (\xi-\varepsilon, \xi+\varepsilon)$, for infinitly many values of n

$\Rightarrow \xi$ is limit pt. of $\{S_n\}$

Hence thm. is proved.

Ex: Does the converse of Bolzano Weierstrass thm holds? Verify it.

Soln: Converse of Bolzano-Weierstrass thm need not be true

Let $\{S_n\} = \{1, 2, 1, 4, 1, 6, \dots\}$

then $\{S_n\}$ has unique limit pt. 1

but $\{S_n\}$ is not bdd above

i.e $\{S_n\}$ is not bdd $\quad (\because 1 \le S_n < \infty)$

IMP Theorem - 5 P.T. the set of the limit pts. of a bounded seq. has the greatest & the least members.

Proof: Let $\{S_n\}$ be any bdd. seq. Let S be its range set then S is bdd and its derived set S' is also bdd.
Let T be the set of limit pts. of seq. $\{S_n\}$ then T is non-empty ($\because$ by B-W thm),

Also W.K.T T contains all limit pts. of S (i.e S') and those pts. of S which are not limit pts of S but are limit pts. of seq. $\{S_n\}$
$\therefore$ T is bounded set.

Case (i) If T is finite, then it has the greatest and the least members
Hence thm. is proved

Case (ii) If T is infinite then T is infinite bdd set
$\therefore$ by the order - completeness property of real nos. we say that T has the supremum M and the infimum m.

Now we p.T. $M \in T$ & $m \in T$.

First p.T $M \in T$.

For $\varepsilon > 0$, Let $(M-\varepsilon, M+\varepsilon)$ be any nbhd. of M

Since M is supremum of T

$\therefore \exists$ atleast one $\xi \in T \ni$

$$M-\varepsilon < \underbrace{\xi \le M}_{} \quad (< M+\varepsilon)$$

Thus $(M-\varepsilon, M+\varepsilon)$ is nbhd. of ξ

But ξ is limit pt of seq $\{S_n\}$

$\therefore$ nbhd $(M-\varepsilon, M+\varepsilon)$ contains infinitely many members of S_n

ie nbhd $(M-\varepsilon, M+\varepsilon)$ of M contains infinitely many members of seq S_n

$\Rightarrow M$ is a limit pt of seq.

$\Rightarrow M \in T$

Similarly we can p.T. $m \in T$

Thus $M \in T$, $m \in T$ are the greatest & the least members of T resp.

Hence Every bdd seq. has the greatest & the least limit pts.

NOTE: (i) The greatest limit pt. of a bdd seq. is called the upper limit or limit superior of the seq. It is denoted by $\lim_{n\to\infty} \sup S_n$.

(ii) The smallest limit pt. of a bdd. seq. is called the lower limit or limit inferior of the seq. It is denoted by

$$\lim_{n\to\infty} \inf S_n .$$

(iii) For bdd seq $\{S_n\}$

if $\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} \inf S_n = l$

then we say that seq $\{S_n\}$ converges to l

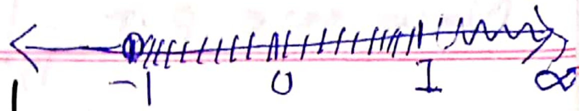
i.e $\lim_{n\to\infty} S_n = l$

(iv) Every bdd. sequence with a unique limit pt is convergent

(v) A limit pt. of the range set of seq. is also a limit pt. of seq. but converse may not be true

M.Imp

Ex: Show that seq. $\{r^n\}$ converges iff $-1 < r \le 1$.



Soln: Case-(i) If $r > 1$

Let $r = 1+h$, $h>0$ then

$$r^n = (1+h)^n$$

$$= 1 + nh + {}^nC_2 h^2 + \cdots + h^n$$

$$= 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots + h^n$$

$$> 1 + nh \quad \forall n \in N$$

Let $G > 0$ be any no however large, we have

$$r^n > 1 + nh > G, \text{ if } n > \frac{G-1}{h}$$

Let $m \in N \ni m > \frac{G-1}{h}$, then

for $G > 0$ $\exists$ +ve integer $m \ni$

$$r^n > G, \quad \forall n \geq m$$

Thus $\lim\limits_{n\to\infty} r^n = \infty$

Hence seq $\{r^n\}$ diverges to ∞.

Case(ii) If $r = 1$, then

$$\{r^n\} = \{1, 1, 1, \ldots \ldots\}$$

$\therefore \lim\limits_{n\to\infty} r^n = 1$

$\therefore$ seq. converges to 1

case-(iii) If $|r|<1$ i.e $-1<r<1$

Now we P.T $\lim_{n\to\infty} r^n = 0$, for $|r|<1$

Let $|r| = \frac{1}{1+h}$, where $h>0$, then

$|r^n| = |r|^n = \frac{1}{(1+h)^n}$

$$= \frac{1}{1+nh+\frac{n(n-1)}{2}h^2+\cdots+h^n}$$

$$< \frac{1}{1+nh} \quad (\because 1+nh+\cdots+h^n > 1+nh)$$

$< \varepsilon$, if $\frac{1}{\varepsilon} < 1+nh$

i.e if $\left(\frac{1}{\varepsilon}-1\right)\frac{1}{h} < n$

Let $m \in N \ni m > \frac{1}{h}\left[\frac{1}{\varepsilon}-1\right]$

~~Then $|r^n|$~~

Thus for $\varepsilon>0$ $\exists$ $m\in N \ni$

$|r^n| < \varepsilon \quad \forall\ n \geq m$

Hence $\lim_{n\to\infty} r^n = 0$

i.e $\{r^n\}$ converges to 0,

when $|r|<1$

case-(iv) If $r=-1$ then

$\{r^n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$

Thus seq. is bdd & it has two limit pts 1 & -1

$\therefore$ seq. $\{r^n\}$ oscillate finitely

case (v): If $r<-1$,

Let $r=-t$, $t>1$ then

$\{r^n\} = \{(-t)^n\} = \{(-1)^n t^n\}$, $(t>1)$

$= \{-t, t^2, -t^3, t^4, -t^5, t^6, \ldots\}$

Thus $-\infty < r^n < \infty$

$\therefore$ seq. is unbounded

$\therefore$ seq. is oscillate infinitely

Hence by case i, ii, iii, iv, v,

we say that

seq. $\{r^n\}$ converges iff $-1<r\leq 1$

<u>Ex:</u> P.T $\{r^n\}$ converges to zero, iff $|r|<1$ i.e $-1<r<1$

<u>sol^n:</u> Write case (iii) of above Ex.

<u>NOTE:-</u> $\{r^n\}$ converges to 0 iff $|r|<1$

M.Imp

Theorem: 6 State and Prove Cauchy's General Principle of Convergence.

Statement: A necessary and sufficient condition for the cgt of a seq. $\{S_n\}$ is that, for each $\varepsilon > 0$ $\exists$ +ve integer $m \ni |S_{n+p} - S_n| < \varepsilon, \forall n \geq m, p \geq 1$.

Proof: Necessary part:-

If $\{S_n\}$ converges to l, then for given $\varepsilon > 0$ $\exists$ +ve integer $m \ni$

$$|S_n - l| < \varepsilon/2 \quad \forall n \geq m$$

If $p \geq 1$, then $n+p > n \geq m$ i.e $n+p \geq m$

$$\therefore |S_{n+p} - l| < \varepsilon/2 \quad \forall n \geq m \text{ \& } p \geq 1$$

Now $|S_{n+p} - S_n| = |S_{n+p} - l + l - S_n|$

$$\leq |S_{n+p} - l| + |l - S_n|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \forall n \geq m \text{ \& } p \geq 1$$

$$= \varepsilon$$

Sufficient part:- If for each $\varepsilon > 0$ $\exists$ +ve integer $m \ni$

$$|S_{n+p} - S_n| < \varepsilon, \quad \forall n \geq m \text{ \& } p \geq 1 \quad \text{(1)}$$

We have to P.T $\{S_n\}$ convergence

First we P.T $\{S_n\}$ is bounded.

By ①, we say that for $\varepsilon = 1$, $\exists$ +ve integer $m_0 \ni$

$$|S_{n+p} - S_n| < 1 \quad \forall\, n \geq m_0 \ \& \ p \geq 1$$

For $n = m_0$, we get

$$|S_{m_0+p} - S_{m_0}| < 1 \quad \forall\, p \geq 1 \qquad \text{②}$$

$$\Rightarrow S_{m_0} - 1 < S_{m_0+p} < S_{m_0} + 1, \ \forall\, p \geq 1$$

Let $g = \min\{s_1, s_2, \dots, S_{m_0-1}, S_{m_0} - 1\}$

$G = \max\{s_1, s_2, \dots, S_{m_0-1}, S_{m_0} + 1\}$

Then $g \leq S_i$ & $G \geq S_i$ $\forall\, i = 1, 2, \dots m_0 - 1$

Also $g \leq S_{m_0} - 1$ & $G \geq S_{m_0} + 1$

$$\Rightarrow g \leq S_i \leq G \quad \forall\, i = 1, 2, \dots m_0 - 1 \qquad \text{③}$$

By ② & ③ we say that

$$g \leq S_n \leq G \quad \forall\, n$$

Hence $\{S_n\}$ is bounded

$\therefore$ By Bolzano-Weierstrass thm. for seq. we say that $\{S_n\}$ has at least one limit pt., say l

Now we P.T $\{S_n\}$ converges to l.

From eqn ①, we say that

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for $\varepsilon > 0$ $\exists$ +ve integer m $\ni$

$|S_{n+p} - S_n| < \varepsilon/3 \quad \forall\, n \geq m \ \& \ p \geq 1$

In particular for $n = m$, we get

$|S_{m+p} - S_m| < \varepsilon/3 \quad \forall\, p \geq 1$

Also l is limit pt. of $\{S_n\}$

$\therefore$ $\exists$ integer $m_1 > m$ $\ni$

$|S_{m_1} - l| < \varepsilon/3$ i.e. $l - \frac{\varepsilon}{3} < S_{m_1} < l + \frac{\varepsilon}{3}$

Since $m_1 > m$, $\therefore$ from (1), we say that $|S_{m_1} - S_m| < \varepsilon/3$

Now $|S_{m+p} - l|$

$= |S_{m+p} - S_m + S_m - S_{m_1} + S_{m_1} - l|$

$\leq |S_{m+p} - S_m| + |S_m - S_{m_1}| + |S_{m_1} - l|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \forall\, p \geq 1$

$= \varepsilon$

Thus $|S_{m+p} - l| < \varepsilon \quad \forall\, p \geq 1$

$\Rightarrow |S_n - l| < \varepsilon \quad \forall\, n \geq m$

Hence $\{S_n\}$ converges to l.

Hence thm. is Proved.

Defn: Cauchy Sequence:-

A seq. $\{S_n\}$ is called a cauchy seq. or a fundamental seq, if for each $\varepsilon > 0$ $\exists$ +ve integer m $\ni$ $|S_{n+p} - S_n| < \varepsilon$, $\forall n \geq m$ & $p \geq 1$

[or $|S_{n_1} - S_{n_2}| < \varepsilon$ $\forall n_1, n_2 \geq m$]

Theorem: 7 P.T a sequence is cgt. iff it is Cauchy seq.

Proof: Write Proof of **Theorem. 6**

Remark: (i) From above thm. we say that a seq. $\{S_n\}$ is not cgt if $\exists$ $\varepsilon > 0$ $\ni$ for every +ve integer m,

$|S_{n+p} - S_n| \not< \varepsilon$, $\forall n \geq m$ & $p \geq 1$

ii) If $\{a_n\}$ & $\{b_n\}$ are two Cauchy seq. then $\{a_n \pm b_n\}$, $\{a_n b_n\}$ & $\left\{\frac{a_n}{b_n}\right\}$ $(b_n \neq 0\ \forall n)$ are also cauchy seq.

iii) Every cauchy seq. is bdd. Converse need not be true.

Ex: show that seq $\{S_n\}$, where $S_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$ can not converge.

Soln. P.T. $\{S_n\}$ is not convergent

i.e P.T $\{S_n\}$ is not cauchy seq.

For $\epsilon = \frac{1}{2}$, $n = m$ & $p = m$ $(m \in N)$

$|S_{n+p} - S_n|$

$= |S_{2m} - S_m|$

$= \left|1+\frac{1}{2}+\dots+\frac{1}{m}+\frac{1}{m+1}+\dots+\frac{1}{2m}-1-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{m}\right|$

$= \left|\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{2m}\right|$

$= \frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{2m}$ $(\because m \in N)$

$> \frac{1}{2m}+\frac{1}{2m}+\dots+\frac{1}{2m}$ $\left(\begin{array}{l}\because m+1<2m \\ m+2<2m \\ \dots\dots\end{array}\right)$

$= \frac{m}{2m} = \frac{1}{2}$

Thus ~~for~~ $|S_{n+p} - S_n| > \frac{1}{2}$

$\therefore \{S_n\}$ is not cauchy seq.

Hence $\{S_n\}$ is not cgt.

Algebra of Sequences

Theorem: 8 If $\{a_n\}$, $\{b_n\}$ be two convergence sequences such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ then P.T.

i) $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$

ii) $\lim_{n\to\infty} (a_n \cdot b_n) = ab$

iii) $\lim_{n\to\infty} \left[\frac{a_n}{b_n} \right] = \frac{a}{b}$, if $b \neq 0$, $b_n \neq 0$, $\forall n$

Proof: (i) Here $\lim a_n = a$, $\lim b_n = b$

$\therefore$ For given $\varepsilon > 0$ $\exists$ $m_1, m_2 \in N \ni$

$|a_n - a| < \varepsilon/2$, $\forall n \geq m_1$

$|b_n - b| < \varepsilon/2$ $\forall n \geq m_2$

Let $m = \max\{m_1, m_2\}$ then

$m \geq m_1$ & $m \geq m_2$

$\therefore$ $|a_n - a| < \varepsilon/2$ & $|b_n - b| < \varepsilon/2$ $\forall n \geq m$

Now $|(a_n \pm b_n) - (a \pm b)|$

$= |(a_n - a) \pm (b_n - b)|$

$\leq |a_n - a| + |b_n - b|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$ $\forall n \geq m$

Hence $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$

(ii) W.K.T $\lim a_n = a$

i.e $\{a_n\}$ is cgt

$\therefore \{a_n\}$ is bdd

$\therefore \exists\ K > 0 \ni |a_n| \leq K \ \forall n$ —— (1)

Now $|a_n b_n - ab|$

$= |a_n b_n - a_n b + a_n b - ab|$

$= |a_n(b_n - b) + b(a_n - a)|$

$\leq |a_n||b_n - b| + |b||a_n - a|$

$\leq K|b_n - b| + |b||a_n - a|$ —— (2)

Let $\varepsilon > 0$ be given.

W.K.T. $\lim a_n = a$ & $\lim b_n = b$

$\therefore \exists\ m_1, m_2 \in N \ni$

$$|a_n - a| < \frac{\varepsilon}{2(|b|+1)} \quad \forall n \geq m_1$$

$$|b_n - b| < \frac{\varepsilon}{2K}, \quad \forall n \geq m_2$$

Let $m = \max\{m_1, m_2\}$ then

$m \geq m_1$ & $m \geq m_2$

$\therefore$ by (2), we say that

$$|a_n b_n - ab| < K\left(\frac{\varepsilon}{2K}\right) + |b| \frac{\varepsilon}{2(|b|+1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \left(\because \frac{|b|}{|b|+1} < 1\right)$$

$$= \varepsilon \quad \forall n \geq m$$

Hence $\lim(a_n b_n) = ab$

(iii) We know that $\lim b_n = b$

First we P.T. $\frac{1}{|b_n|} < \frac{2}{|b|}$

Since $\lim b_n = b$

$\therefore$ For $\varepsilon = \frac{|b|}{2} > 0 \ \exists \ m_1 \in N \ni$

$|b_n - b| < \frac{|b|}{2} \quad \forall \ n \geq m_1$

$\Rightarrow |b| - |b_n| \leq |b_n - b| < \frac{|b|}{2}$

$(\because |b_n - b| \geq \big| |b_n| - |b| \big|)$

$\Rightarrow -|b_n| < \frac{|b|}{2} - |b|$

$\Rightarrow -|b_n| < -\frac{|b|}{2}$

$\Rightarrow |b_n| > \frac{|b|}{2}$

$\Rightarrow \frac{1}{|b_n|} < \frac{2}{|b|} \quad \forall \ n \geq m_1$ —— (1)

Now $\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \left| \frac{b a_n - a b_n}{b \, b_n} \right|$

$= \left| \frac{b a_n - ab + ab - a b_n}{b \, b_n} \right|$

$= \frac{|b(a_n - a) + a(b - b_n)|}{|b| \, |b_n|}$

$\leq \frac{|a_n - a|}{|b_n|} + \frac{|a| \, |b - b_n|}{|b| \, |b_n|}$

$$< \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| \quad \forall n \geq m_1$$

(by (1)) — (2)

Let $\varepsilon > 0$ be given

Since $\lim a_n = a$ & $\lim b_n = b$

$\therefore \exists\, m_2, m_3 \in \mathbb{N} \ni$

$$|a_n - a| < \frac{|b|\varepsilon}{4} \quad \forall n \geq m_2 \quad \text{— (3)}$$

$$|b_n - b| < \frac{|b|^2 \varepsilon}{4(|a|+1)} \quad \forall n \geq m_3 \quad \text{— (4)}$$

Let $m = \max\{m_1, m_2, m_3\}$, then by (2), (3) & (4), we say that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \cancel{< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n \geq m}$$

$$< \frac{2}{|b|} \times \frac{|b|\varepsilon}{4} + \frac{2|a|}{|b|^2} \times \frac{|b|^2\varepsilon}{4(|a|+1)}, \quad \forall n \geq m$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \left(\because \frac{|a|}{|a|+1} < 1\right) \quad \forall n \geq m$$

$$= \varepsilon$$

Hence $\lim \frac{a_n}{b_n} = \frac{a}{b}$

Remark: (i) From above theorem we say that the sum, difference, Product & quotient of two cgt. seq. is also cgt. But converse need not be true.

(ii) If the seq. $\{a_n \pm b_n\}$, $\{a_n b_n\}$ or $\left\{\frac{a_n}{b_n}\right\}$ is cgt then the seq. $\{a_n\}$ & $\{b_n\}$ may not be cgt.

(iii) For example consider seq. $\{a_n\}$ and $\{b_n\}$ where

(I) $a_n = n^2$, $b_n = -n^2$

then seq. $\{a_n + b_n\}$ converges to 0 & seq. $\left\{\frac{a_n}{b_n}\right\}$ converges to -1

but $\{a_n\}$ & $\{b_n\}$ are divergent.

(II) If $a_n = b_n = (-1)^n$ then

seq. $\{a_n - b_n\}$ cgt. to 0,

seq. $\{a_n/b_n\}$ cgt. to 1 &

seq. $\{a_n b_n\}$ cgt to 1

but $\{a_n\}$ & $\{b_n\}$ are oscillate finitely

(III) If $a_n = (-1)^n$, $b_n = (-1)^{n+1}$ then seq. $\{a_n + b_n\}$ cgt to 0, seq $\{a_n b_n\}$ cgt to -1 & $\{a_n/b_n\}$ cgt to -1 but $\{a_n\}$ & $\{b_n\}$ are not cgt.

Ex: show that $\lim_{n \to \infty} \frac{(3n+1)(n-2)}{n(n+3)} = 3$

LHS $= \lim_{n \to \infty} \frac{n^2[3+1/n][1-2/n]}{n^2[1+3/n]}$

$= \frac{(3+0)(1-0)}{1} = 3 = $ RHS

Theorem: 9 If $\lim a_n = a$ & $a_n \geq 0$, $\forall n$, then P.T. $a \geq 0$

Proof: Suppose $a < 0$ then $-a/2 > 0$

Let $\varepsilon = -a/2 > 0$, since $\lim a_n = a$

$\therefore \exists$ +ve integer $m \ni$

$|a_n - a| < \frac{-a}{2}$, $\forall n \geq m$

$\Rightarrow a + \frac{a}{2} < a_n < a - a/2 \quad \forall n \geq m$

$\Rightarrow \frac{3a}{2} < a_n < \frac{a}{2} \quad \forall n \geq m$

$\Rightarrow a_n < \frac{a}{2} < 0$

$\Rightarrow a_n < 0 \quad \forall n \geq m$

$\times$. ($\because a_n \geq 0 \ \forall n$)

$\therefore$ our supposition is wrong.

Hence $a \geq 0$

Thm: 10 If $\{a_n\}$, $\{b_n\}$ are two seq. such that

(i) $a_n \leq b_n \ \forall n$

(ii) $\lim a_n = a$, $\lim b_n = b$

then P.T $a \leq b$

Proof: We K.T. $a_n \le b_n \ \forall n$

$\therefore b_n - a_n \geq 0 \ \forall n$

Also $\lim (b_n - a_n) = b - a$

$\therefore$ by above thm. we say that

$b - a \geq 0 \ \therefore b \geq a$ i.e. $a \le b$

$\left[\because \text{ If } \lim a_n = a \ \& \ a_n \geq 0 \ \forall n \text{ then } a \geq 0 \right]$ (Prove it)

Theorem: 11 State and Prove Sandwich theorem

Statment: ~~Proof:~~ If $\{a_n\}, \{b_n\}, \{c_n\}$ are three seq. such that

i) $a_n \le b_n \le c_n \ \forall n$

ii) $\lim a_n = \lim c_n = l$

then P.T $\lim b_n = l$

Proof: Let $\varepsilon > 0$ be given

Since $\lim a_n = l$, $\lim c_n = l$

$\therefore \exists \ m_1, m_2 \in N \ni$

$$\left.\begin{aligned} |a_n - l| < \varepsilon \quad & \forall n \geq m_1 \\ |c_n - l| < \varepsilon \quad & \forall n \geq m_2 \end{aligned}\right\} \text{— (1)}$$

Let $m = \max\{m_1, m_2\}$ then $m \geq m_1$ & $m \geq m_2$

$$\text{Also } \left.\begin{aligned} l - \varepsilon < a_n < l + \varepsilon, \\ l - \varepsilon < c_n < l + \varepsilon \end{aligned}\right\} \text{(by (1))}$$

& $a_n \le b_n \le c_n$ (given)

$\Rightarrow l-\epsilon < a_n \leq b_n \leq c_n < l+\epsilon \quad \forall n \geq m$

$\Rightarrow l-\epsilon < b_n < l+\epsilon, \quad \forall n \geq m$

$\Rightarrow |b_n - l| < \epsilon, \quad \forall n \geq m$

$\Rightarrow \lim b_n = l$

Ex: Show that seq $\{b_n\}$ converges to 0, where $b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}\right]$

Soln: W.K.T

$$b_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}$$

$$\leq \frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2} \quad \left(\because \begin{array}{l} n+1 > n \\ n+2 > n \dots \end{array}\right)$$

$$= \frac{n}{n^2} = \frac{1}{n}$$

Thus $b_n \leq \frac{1}{n} \quad \forall n$ ——— (1)

Also $b_n \geq \frac{1}{(n+n)^2} + \frac{1}{(n+n)^2} + \dots + \frac{1}{(2n)^2}$

$$= \frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \dots + \frac{1}{(2n)^2} \quad \left[\because \begin{array}{l} n+1 < n+n \\ n+2 < n+n \\ \dots\dots \end{array}\right.$$

$$= \frac{n}{(2n)^2} = \frac{1}{4n}$$

Thus $b_n \geq \frac{1}{4n} \quad \forall n$ ——— (2)

by (1) & (2) $\frac{1}{4n} \leq b_n \leq \frac{1}{n} \quad \forall n$

$\Rightarrow a_n \leq b_n \leq c_n \quad \forall n$, where

$a_n = \frac{1}{4n}, \quad c_n = \frac{1}{n}$

Also $\lim a_n = \lim c_n = 0$

Thus $a_n \le b_n \le c_n$ &

$\lim a_n = \lim c_n = 0$

$\therefore$ by above thm. $\lim b_n = 0$

<u>Theorem : 12</u> State & Prove Cauchy's first theorem on limits.

Statement: If $\lim\limits_{n\to\infty} a_n = l$ then P.T.

$$\lim_{n\to\infty}\left[\frac{a_1+a_2+\cdots+a_n}{n}\right] = l$$

Proof: Let $b_n = a_n - l$, then

$\lim b_n = \lim a_n - l = l - l = 0$

$\therefore \lim\limits_{n\to\infty} b_n = 0$

$\Rightarrow \{b_n\}$ is cgt

$\Rightarrow \{b_n\}$ is bdd

$\therefore \exists\ K > 0 \ni |b_n| < K \ \forall n$ —— ①

Also $\lim b_n = 0$

$\therefore$ For given $\varepsilon > 0\ \exists\ m \in N \ni$

$|b_n| < \varepsilon/2 \quad \forall\ n \ge m$ —— ②

Also $\left|\dfrac{b_1+b_2+\cdots+b_n}{n}\right|$

$$= \left|\frac{b_1+b_2+\cdots+b_m}{n} + \frac{b_{m+1}+\cdots+b_n}{n}\right|$$

$$\le \frac{|b_1|+|b_2|+\cdots+|b_m|}{n} + \frac{|b_{m+1}|+\cdots+|b_n|}{n}$$

$$< \frac{1}{n}[K+\cdots+K] + \frac{1}{n}\left[\frac{\varepsilon}{2}+\cdots+\frac{\varepsilon}{2}\right]$$

(by ① & ②)

$$= \frac{1}{n}(mK) + \frac{1}{n}\left\{(n-m)\frac{\varepsilon}{2}\right]$$

$$= \frac{mK}{n} + \frac{\varepsilon}{2}\left(\frac{n-m}{n}\right)$$

$$< \frac{mK}{n} + \underbrace{\frac{\varepsilon}{2}}_{(3)} \quad \left(\because n-m < n \quad \therefore \frac{n-m}{n} < 1\right)$$

Let $m_1 \in \mathbb{N} \ni m_1 > \frac{2mK}{\varepsilon}$

then $n \geq m_1 \Rightarrow n > \frac{2mK}{\varepsilon}$

$$\Rightarrow \frac{mK}{n} < \frac{\varepsilon}{2}$$

Thus $\frac{mK}{n} < \frac{\varepsilon}{2} \quad \forall\, n \geq m_1$

Let $m' = \max\{m, m_1\}$, then

for $n > m'$, we have

$$\left|\frac{b_1+b_2+\cdots+b_n}{n}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \quad (\text{by } (3))$$

Thus

$$\lim_{n\to\infty} \frac{b_1+b_2+\cdots+b_n}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} - l = 0$$

$(\because b_n = a_n - l \ \forall n)$

$$\Rightarrow \lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l$$

Ex: Does the converse of above thm. true! verify it.

Sol^n: Converse of above thm. need not be true

Let $a_n = (-1)^n$ then

$$\frac{a_1+a_2+\dots+a_n}{n} = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{1}{n}, & \text{if } n \text{ is odd} \end{cases}$$

Thus $\lim \dfrac{a_1+a_2+\dots+a_n}{n} = 0$

but seq. $\{a_n\}$ is not cgt.

$(\because \{a_n\} = \{-1, 1, -1, 1, \dots\})$

Ex: P.T. $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right] = 1$

Sol^n: Let $a_n = \dfrac{n}{\sqrt{n^2+n}}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+1/n}} = \frac{1}{\sqrt{1+0}} = 1$$

$\therefore$ By Cauchy's first thm. we say that

$$\lim \frac{a_1+a_2+\dots+a_n}{n} = 1$$

$$\Rightarrow \lim \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + \dots + \frac{n}{\sqrt{n^2+n}}\right] = 1$$

$$\Rightarrow \lim \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right] = 1$$

Ex. S.T. $\lim_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/3}+\dots+n^{1/n}\right]=1$

Soln: Let $a_n = n^{1/n}$, then

$\lim a_n = \lim n^{1/n} = 1$

$\therefore$ By Cauchy's first thm on limits, we say that

$$\lim_{n\to\infty} \frac{a_1+a_2+\dots+a_n}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/3}+\dots+n^{1/n}\right]=1$$

Theorem: 13 If $\{a_n\}$ be a seq. such that $\lim \frac{a_{n+1}}{a_n} = l$, where $|l|<1$ then

P.T. $\lim_{n\to\infty} a_n = 0$.

Proof: since $|l|<1$, $\therefore$ we can choose a +ve no. ε so small $\ni |l|+\varepsilon<1$

W.K.T $\lim \frac{a_{n+1}}{a_n} = l$

$\therefore \exists\, m \in N \ni \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon, \ \forall n \geq m$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| - |l| \leq \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < |l|+\varepsilon = K \text{ say, where } K<1 \quad \text{--- (1)} \quad \forall n \geq m$$

Put $n = m, m+1, \ldots n-1$ & multiplying we get

$$\left|\frac{a_{m+1}}{a_m}\right| \left|\frac{a_{m+2}}{a_{m+1}}\right| \cdot \left|\frac{a_{m+3}}{a_{m+2}}\right| \cdots \left|\frac{a_n}{a_{n-1}}\right|$$

$< K \cdot K \cdot K \cdots K$ ($n-m$ times)

$$\Rightarrow \left|\frac{a_n}{a_m}\right| < K^{n-m}$$

$$\Rightarrow |a_n| < \frac{|a_m|}{K^m} K^n \quad \forall n \geq m$$

(but $0 < K < 1$ $\therefore K^n \to 0$)

~~Hence $\lim_{n\to\infty} a_n = 0$~~

where $\frac{|a_m|}{K^m} > 0$

$\Rightarrow$ $0 < \lim |a_n| < 0 \Rightarrow \lim |a_n| = 0$

Hence $\lim_{n\to\infty} a_n = 0$

Theorem: 14 If $\{a_n\}$ be a seq. such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1$ then P.T.

$$\lim_{n\to\infty} a_n = \infty$$

Proof: Since $l > 1$ $\therefore$ we can choose a +ve no. $\varepsilon \ni l - \varepsilon > 1$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

$\therefore \exists\, m \in N \ni$

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon, \quad \forall n \geq m$$

$\Rightarrow l-\varepsilon < \frac{a_{n+1}}{a_n} < l+\varepsilon \quad \forall n \geq m$

$\Rightarrow \frac{a_{n+1}}{a_n} > l-\varepsilon = K$ say, where $K>1$

Putting $n = m, m+1, \ldots n-1$ & multiplying we get

$$\frac{a_{m+1}}{a_m} \cdot \frac{a_{m+2}}{a_{m+1}} \cdot \ldots \ldots \cdot \frac{a_n}{a_{n-1}} > K \cdot K \ldots K$$

$$\Rightarrow \frac{a_n}{a_m} > K^{n-m}$$

$$\Rightarrow \left|\frac{a_n}{a_m}\right| \geq \frac{a_n}{a_m} > K^{n-m}$$

$$\Rightarrow |a_n| > \frac{|a_m|}{K^m} K^n \rightarrow \infty \quad \left(\because K>1 \;\therefore K^n \rightarrow \infty\right)$$

$$\Rightarrow \lim_{n\to\infty} a_n = \infty \quad \left[\because \frac{|a_m|}{K^m} > 0 \;\&\; K^n \rightarrow \infty\right]$$

Ex. P.T $\lim \frac{x^n}{n!} = 0$, for any $x \in R$

Soln. Let $a_n = \frac{x^n}{n!}$, then $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \frac{x}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$$

$\therefore$ by above thm we say that $\lim_{n\to\infty} a_n = 0$ i.e $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Ex. P.T

$$\lim_{n\to\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n = 0, \quad |x|<1$$

Sol^n:

Let $a_n = \dfrac{m(m-1)\cdots\cdots(m-n+1)}{n!} x^n$

then

$$a_{n+1} = \frac{m(m-1)\cdots(m-(n-1))(m-n)}{(n+1)!} x^{n+1}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(m-n)}{(n+1)} x$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{m-n}{n+1}\right) x$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left[\frac{m}{n}-1\right]}{\cancel{n}\left[1+\frac{1}{n}\right]} x$$

$$= \left(\frac{0-1}{1+0}\right) x = -x$$

But $|-x| = |x| < 1$

$\therefore$ by thm. $\lim_{n\to\infty} a_n = 0$

Hence

$$\lim_{n\to\infty} \frac{m(m-1)\cdots\cdots(m-n+1)}{n!} x^n = 0$$

Monotonic Sequences

Def^n: seq. $\{S_n\}$ is said to be

i) monotonic increasing if
$S_{n+1} \geq S_n \ \forall n$ (i.e. $S_1 \leq S_2 \leq S_3 \leq \ldots$)

ii) monotonic decreasing if
$S_{n+1} \leq S_n \ \forall n$ (i.e. $S_1 \geq S_2 \geq S_3 \geq S_4 \geq \ldots$)

iii) monotonic if it is either monotonic increasing or monotonic decreasing

iv) strictly increasing if
$S_{n+1} > S_n \ \forall n$

v) Strictly decreasing if
$S_{n+1} < S_n \ \forall n$

Remark: 1) Monotonic sequences are either cgt or divergent, they can not oscillate.

Theorem: 15 A necessary & sufficient condition for the convergence of a monotonic seq. is that it is bounded.

Proof: Necessary condition:-
If monotonic seq. is cgt. then it is bounded
($\because$ W.K.T. every cgt. seq. is bdd)

Sufficient Part: If monotonic seq. is bounded

Let $\{S_n\}$ be any monotonic increasing & bounded

Let S be the range of $\{S_n\}$ then S is also bounded

$\therefore$ By order completeness Property, we say that S has the supremum say M.

Now we P.T $\{S_n\}$ converges to M.

Let $\varepsilon > 0$ be given

since M is sup. of $\{S_n\}$

$\therefore \exists$ at least one $S_m \ni$

$M - \varepsilon < S_m \leq M$

Also W.K.T $\{S_n\}$ is monotonic increasing seq.

$\therefore S_n \geq S_m \quad \forall\, n \geq m$

$\Rightarrow S_n \geq S_m > M - \varepsilon \quad \forall\, n \geq m$

since M is sup.

$\therefore S_n \leq M < M + \varepsilon \quad \forall\, n$

Thus $M - \varepsilon < S_n < M + \varepsilon \quad \forall\, n \geq m$

$\Rightarrow |S_n - M| < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow \lim S_n = M$

$\Rightarrow \{S_n\}$ is conv.

Similarly we can prove for bdd. monotonic decreasing seq.

Hence thm. is Proved

Remark: A monotonic increasing bdd. above seq. is converges to its supremum (14b) and a monotonic decreasing bdd. below seq. is converges to its infimum (9.1.6)

Thm: 16 P.T. every monotonic increasing seq. which is not bdd above, diverges to $+\infty$.

Proof: Let $\{S_n\}$ be any mono. increasing seq. which is not bdd. above

P.T $\lim S_n = +\infty$

Let $G > 0$ be any real no. however large

Since $\{S_n\}$ is not bdd above

$\therefore \exists\ m \in N \ni S_m > G$

Also $\{S_n\}$ is mono. increasing

$\therefore S_n \geq S_m \quad \forall\ n \geq m$

Thus $S_n > G \quad \forall\ n \geq m$

$\Rightarrow \lim_{n\to\infty} S_n = +\infty$

Hence $\{S_n\}$ diverges to $+\infty$.

Def^n: <u>Subsequences:</u>

If $\{n_k\}$ is strictly mono. increasing seq. of natural nos. i.e $n_1 < n_2 < n_3 < \cdots$, then $\{S_{n_k}\}$ is called a subsequence of $\{S_n\}$.

e.g (i) $\{S_2, S_4, \ldots S_{2n}, \ldots\}$ is a sub seq. of $\{S_n\}$

(ii) $\{S_1, S_4, S_9, \ldots\}$ is a subseq. of $\{S_n\}$

(iii) $\{S_5, S_6, S_7, \ldots\}$ is a subseq. of $\{S_n\}$.

<u>Remark:</u>

i) $\{S_n\}$ cgt. to S iff its every sub seq. cgt. to S.

→ $\{S_n\}$ divt. to $+\infty$ (or $-\infty$) iff every subseq. of $\{S_n\}$ divt. to $+\infty$ (or $-\infty$)

ii) If ξ is a limit pt. of seq. $\{S_n\}$ then $\exists$ a subseq. of $\{S_n\}$ which converges to ξ.

Ex: S.T. the seq. $\{S_n\}$, where $S_n = \left[1+\frac{1}{n}\right]^n$, is cgt. ~~to~~ and $\lim \left[1+\frac{1}{n}\right]^n$ lies between 2 & 3.

Soln: We know that

$$S_n = \left[1+\frac{1}{n}\right]^n$$

$$= 1 + {}^nC_1 \frac{1}{n} + {}^nC_2 \frac{1}{n^2} + \cdots + {}^nC_n \frac{1}{n^n}$$

$$= 1 + 1 + \frac{n(n-1)}{2!} \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 3.2.1}{n!} \times \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$

$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Since each term after first two term on the RHS is increasing fn of n

$\therefore \{S_n\}$ is a mono. increasing seq.

Also each bracket on RHS is positive

$\therefore S_n > 2$ & $S_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$\Rightarrow 2 < S_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 3$

$\Rightarrow 2 < S_n < 3 \quad \forall n$

Thus $\{S_n\}$ is bdd & mono. increasing

$\therefore \{S_n\}$ is convergent

Also $\lim S_n = e$, where $2<e<3$

ie $\lim_{n\to\infty} \left[1+\frac{1}{n}\right]^n = e$, where $2<e<3$

EX: P.T $\{S_n\}$ is cgt. where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

Sol^n: W.K.T $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$$\therefore S_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$\Rightarrow S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{2n+2+2n+1-2(2n+1)}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0$$

$\Rightarrow S_{n+1} - S_n > 0 \quad \forall n$

$\Rightarrow S_{n+1} > S_n \quad \forall n$

$\therefore \{S_n\}$ is mono. increasing seq.

Also $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \quad \left(\begin{array}{l} \because n+1>n \\ n+2>n \\ \cdots\cdots \end{array}\right)$$

$$= \frac{n}{n} = 1$$

$\Rightarrow S_n < 1$

Thus $0 < S_n < 1$ $\left(\because S_n > 0 \ \forall n\right)$

Thus $\{S_n\}$ is bdd

Thus $\{S_n\}$ is bdd. & mono. increasing

$\therefore$ by thm. $\{S_n\}$ is convergent.

Ex: S.T $\{S_n\}$ is convergent, where

$$S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Soln: W.K.T $S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$\therefore S_{n+1} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$\Rightarrow S_{n+1} - S_n = \frac{1}{(n+1)!} > 0, \ \forall n$

$\Rightarrow S_{n+1} > S_n \quad \forall n$

$\Rightarrow \{S_n\}$ is mono. increasing

Also $S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

$(\because 3! > 2^2, \ 4! > 2^3 \ldots\ldots)$

$= \dfrac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$ $\left[\because 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}, \text{ if } r < 1\right]$

$= \dfrac{1 - \frac{1}{2^n}}{\frac{1}{2}} = 2\left[1 - \frac{1}{2^n}\right] < 2$

$\Rightarrow S_n < 2$

$\Rightarrow 0 < S_n < 2$

$\therefore \{S_n\}$ is bdd

Thus $\{S_n\}$ is bdd & mono. increasing

$\therefore \{S_n\}$ is convergent

Ex: Show that $\{S_n\}$, defined by the recursion formula

$S_{n+1} = \sqrt{3S_n}$, $S_1 = 1$,

converges to 3.

<u>Soln</u>: Here the terms of seq $\{S_n\}$ are $1, \sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots$

clearly $S_1 < S_2 < S_3 < \ldots S_n < \ldots$

$\therefore \{S_n\}$ is mono. increasing

Also $S_1 < 3$, $S_2 < 3$, $S_3 = \sqrt{3S_2} < 3, \ldots$

$S_m < 3 \Rightarrow \sqrt{3S_m} < \sqrt{3 \cdot 3} \Rightarrow S_{m+1} < 3$

Thus $S_n < 3 \quad \forall n$

$\therefore 0 < S_n < 3, \forall n$

Thus $\{S_n\}$ is bdd & mono. incre.

$\therefore \{S_n\}$ is cgt.

Let $\lim S_n = l$, then

$\lim S_{n+1} = l$

$\Rightarrow \lim \sqrt{3S_n} = l$

$\Rightarrow \sqrt{3l} = l \quad (\because \lim S_n = l)$

$\Rightarrow 3l = l^2$

$\Rightarrow l(l-3) = 0$

$\Rightarrow l = 0$ or $l = 3$

but $l \neq 0$ $(\because S_n \geq 1 \ \forall n)$

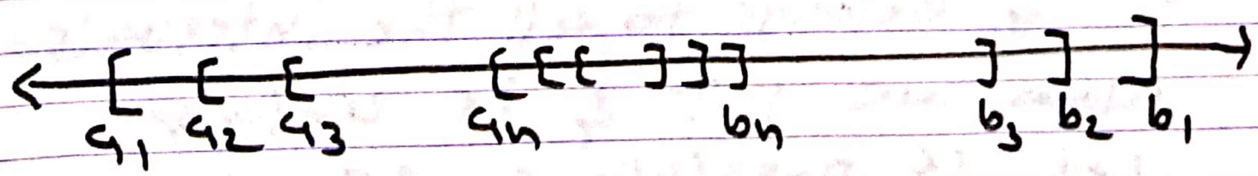
$\therefore l = 3$

Hence $\lim\limits_{n \to \infty} S_n = 3$

Theorem: 17 State & Prove Nested-interval Theorem.

~~P~~ Statement:- If a seq. of closed intervals $[a_n, b_n]$ is such that each member $[a_{n+1}, b_{n+1}]$ is contained in the preceding one $[a_n, b_n]$ and $\lim(b_n - a_n) = 0$ then prove that there one & only one pt. common to all the intervals of the sequence.

Proof: Here we have given that



clearly $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$

$b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq \cdots$

Thus $\{a_n\}$ is mono. increasing &
$\{b_n\}$ is mono. decreasing

Also $a_n \leq b_1 \quad \forall n$ &
$b_n \geq a_1 \quad \forall n$

$\therefore$ $\{a_n\}$ is bdd. above by b_1 &
$\{b_n\}$ is bdd. below by a_1

$\therefore$ both seq. $\{a_n\}$ & $\{b_n\}$ are cgt.

Let $\lim a_n = \xi$ & $\lim b_n = \eta$

W.K.T

$0 = \lim(b_n - a_n)$

$\Rightarrow 0 = \lim b_n - \lim a_n = \xi - \eta$

$\Rightarrow \xi = \eta$

obviously ξ is the upper bound of $\{a_n\}$ & is the lower bound of $\{b_n\}$

Thus $a_n \leq \xi \leq b_n \quad \forall n$

i.e $\xi \in [a_n, b_n] \quad \forall n$

i.e ξ belongs to all the intervals

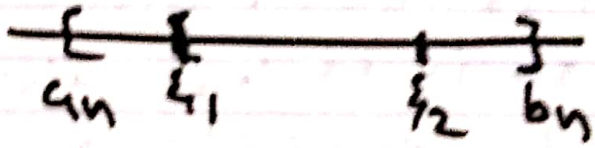
Now we P.T ξ is unique

Let, if possible ξ_1 & ξ_2 be two different pt. common to all the intervals & let $\xi_1 < \xi_2$

Then $\xi_1 \in [a_n, b_n]$, $\xi_2 \in [a_n, b_n]$

Thus $a_n \leq \xi_1 < \xi_2 \leq b_n \quad \forall n$

$\Rightarrow b_n - a_n \geq \xi_2 - \xi_1 \neq 0 \quad \forall n$



$\Rightarrow \lim (b_n - a_n) \geq \xi_2 - \xi_1$

$\Rightarrow \quad 0 \geq \xi_2 - \xi_1$

$\Rightarrow \quad \xi_1 \geq \xi_2 \quad \cdot\!\times\!\cdot \quad (\because \xi_1 < \xi_2)$

$\therefore$ our supposition is wrong
Hence $\xi_1 = \xi_2$

Hence result is proved

Ex. Which of following seq. are mono. increasing! Also find its cgt.

i) $\{\frac{1}{n}\}$

Sol: $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$\therefore$ It is mono. decreasing Also it is bdd below $\because 0 < S_n \leq 1$
& $\lim \inf \{\frac{1}{n}\} = 0$

$\therefore \lim_{n \to \infty} \{\frac{1}{n}\} = 0$

ii) $\left\{ \frac{5n-1}{n} \right\}$

Sol^n $\left\{ \frac{5n-1}{n} \right\} = \left\{ 4, \frac{9}{2}, \frac{14}{3}, \cdots\cdots \right\}$

It is mono. increasing

Also it is bdd above [$\because 4 \le S_n < 5$]

$\therefore \lim\sup \left(\frac{5n-1}{n} \right) = 5$

$\therefore \lim \left(\frac{5n-1}{n} \right) = 5$

6) P.T $\lim \frac{n^2+1}{n^2} = 1$

For any $\varepsilon > 0$

$$\left| \frac{n^2+1}{n^2} - 1 \right| = \left| \frac{n^2+1-n^2}{n^2} \right| = \left| \frac{1}{n^2} \right|$$

$$= \frac{1}{n^2} < \varepsilon, \text{ if } n > \frac{1}{\sqrt{\varepsilon}}$$

Let $m \in \mathbb{N} \ni m > \frac{1}{\sqrt{\varepsilon}}$

Thus $\left| \frac{n^2+1}{n^2} - 1 \right| < \varepsilon \ \forall \ n \geq m$

$\therefore \lim \frac{n^2+1}{n^2} = 1$

Ex. If $\lim x_n = l$ then P.T. $\lim |x_n| = |l|$. Does the converse hold? Verify it.

Soln: For any $\varepsilon > 0$

since $\lim x_n = l$

$\therefore \exists \ m \in N \ni |x_n - l| < \varepsilon \ \forall \ n \geq m$

$\Rightarrow \big| |x_n| - |l| \big| \leq |x_n - l| < \varepsilon \ \forall n \geq m$

$\Rightarrow \big| |x_n| - |l| \big| < \varepsilon, \ \forall \ n \geq m$

$\Rightarrow \lim_{n \to \infty} |x_n| = |l|$

Converse need not be true

If $x_n = (-1)^n$ then

$|x_n| = |(-1)^n| = \{1, 1, 1, 1, \ldots\}$

$\therefore \lim |x_n| = 1$

but $\{x_n\} = \{-1, 1, -1, 1, \ldots\}$

$\therefore \lim_{n \to \infty} x_n$ does not exist

$\therefore$ converse is not true.

Def 1: Null seq.: seq. $\{s_n\}$ is said to be null seq. if $\{s_n\}$ converges to 0.

e.g $\{\frac{1}{n^2}\}$ is null seq.

Ex: P.T $\{nq^n\}$ where $|q|<1$ is a null seq.

Sol^n: If $q=0$ then $\{nq^n\} = \{0,0,---\}$

$\therefore \{nq^n\}$ is cgt. to 0.

If $q \neq 0$ & $|q|<1$

Let $|q| = \frac{1}{1+h}$, $h>0$, then

$$|nq^n| = \frac{n}{(1+h)^n}$$

$$= \frac{n}{1+nh+\frac{n(n-1)}{2}h^2+---+h^n}$$

$$< \frac{n}{\frac{n(n-1)}{2}h^2} = \frac{2}{(n-1)h^2} < \varepsilon$$

$$\text{if } \frac{2}{h^2\varepsilon} < n-1$$

$$\text{i.e. } n > \frac{2}{h^2\varepsilon}+1$$

Let $m \in N \ni m > \frac{2}{h^2\varepsilon}+1$,

Hence $\lim_{n\to\infty}(nq^n) = 0$

$\therefore$ it is null seq.